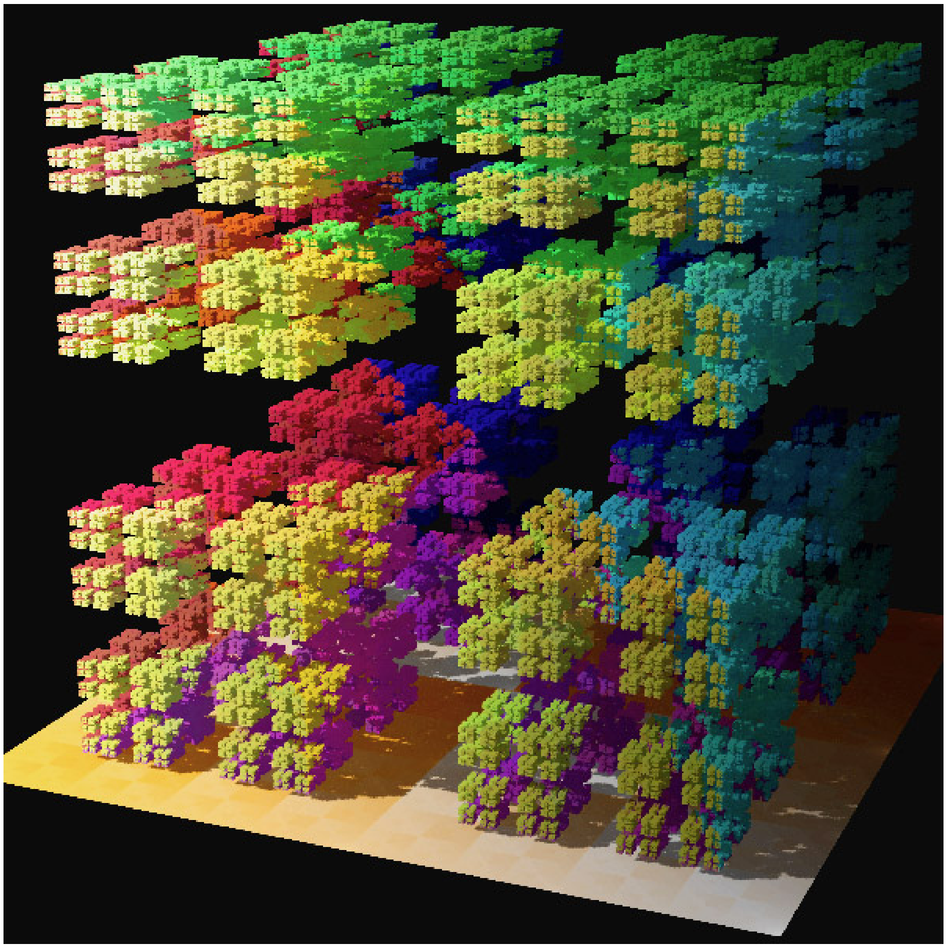

Cantor Set

The next fractal you will build is known as the *Cantor set* after **Georg Cantor** (German mathematician; 1845 - 1918) whose pioneering work on the infinite was transformative and nicely complements the development of fractals:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not - I do not say divisible - but actually divisible; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

Interestingly, the Cantor set was only rediscovered by Cantor, it had been previously invented by **H.J.S. Smith** (Irish mathematician; 1826 - 1883) more than 10 years early. Unfortunately, his contribution went unnoticed until much more recently.[1]

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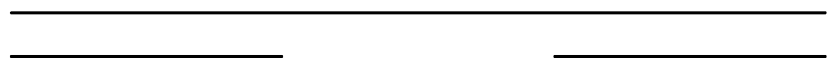
[1] See "A History of the Cantor Set and Cantor Function" by Julian F. Fleron, *Mathematics Magazine*, vol. 67, no. 2, April 1994, pp. 136-40.



Initiator (top) and generator (bottom) for the Cantor set.

The **Cantor set** is the self-similar fractal formed from an initiator that is a line segment of unit length and a generator that is comprised of two line segments of length one-third and which are separated by an empty space one-third of a unit wide - as shown above.

A scale copy of the Cantor set initiator and generator, with nicely sized rule lines, is included on the last page of this lesson.

1. Create Stage 2 of the Cantor set.

2. How many line segments make up Stage 2? How long is each line segment (in terms of the initiator which is 1 unit long)? So what is the overall “length” of Stage 2 of the Cantor set? **102.**

3. Create Stage 3 of the Cantor set.

4. How many line segments make up Stage 3? How long is each line segment? So what is the overall “length” of Stage 3 of the Cantor set?

5. Create Stage 4 of the Cantor set.

6. How many line segments make up Stage 4? How long is each line segment? So what is the overall “length” of Stage 4 of the Cantor set?

7. Find an expression for the “length” of the Cantor set in Stage *n*.

8. Return to the Initiator and Generator. Explain how interpret your expression for the length of Stage *n* of the Cantor set in terms of the geometry of the Initiator and Generator.

9. What seems to be happening as we continue this process? What will the “length” of the Cantor set will be?

The Cantor set does seem to disappear in front of our eyes as we try to generate it. Many have called it “Cantor dust” for this reason. And while it does have *measure* zero it has not disappeared completely. At each stage we only remove the *open* middle third so the endpoints of each copy of the initiator will forever remain. If the initiator was built on a standard number line so its endpoints were $x = 0$ and $x = 1$ then the endpoints $x = \frac{1}{3}$ and $x = \frac{2}{3}$ of the copies of the Initiator that make up the Generator will never be removed; i.e. are points in the Cantor set.

And after Stage 2 the points $x = \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}$ will be be endpoints adjacent to middle thirds that have been removed. Hence, they will be in the Cantor set. And, in fact, it is not just points like this that are in the Cantor set. There are exactly as many points in the Cantor set as there are in the whole number line - an amazing discovery of Cantor. In terms of physical size the Cantor set is vanishingly small, in terms of how numerous its elements are it is as large as the line it was first created from![6] So how can one object be so sparse in some sense and so enormous in another? Perhaps because we are viewing it from the confines of our "limited dimensionality" that A Square spoke of in Flatland.

The Cantor set was born from an Initiator and Generator that were one-dimensional. While it is not one-dimensional, it is far from the zero-dimensional inhabitants of Pointland.

10. To construct the Cantor set begin by drawing a line segment of unit length.

11. Divide this segment into thirds. You are to remove the middle third, leaving the endpoints $\frac{1}{3}$ and $\frac{2}{3}$.

12. Draw a picture of what remains after this first step.

13. Divide each of the line segments that remain from Investigation **12** into thirds. Delete each of the middle thirds, again leaving the endpoints. Draw a picture of what remains after this second step.

14. Divide each of the line segments that remain from Investigation **13** into thirds. Delete each of the middle thirds, again leaving the endpoints. Draw a picture of what remains after this third step.

15. Draw a picture of what remains after the fourth step in this construction.

The **Cantor set** is the set of points that are never removed from the unit interval when this process is continued indefinitely.

16. Find a number of points that are in the Cantor set. How many such points are there?

17. One can show that the point $\frac{1}{4}$ is not an endpoint of any of the removed intervals. Nonetheless, it is a member of the Cantor set. Does this surprise you? Why?

18. What is the linear scaling factor for the Cantor set?

19. Determine the self-similarity dimension of the Cantor set.